Self-Organized Criticality II → Hydrodynamic Models

Recall

→ SOC idea
→ Sandpile Model (CA)

Now, natural to ask:

- is there a continuum model, as $l_{avalanche} \gg \Delta$?

Can one think in terms of avalanche turbulence

- can one exploit symmetry in deriving SOC model, much as symmetry exploited in Ginzburg-Landau model

These bring us to the hydrodynamic theory/model of SOC.

→ continuum model

→ valid for large scales, long time scales.

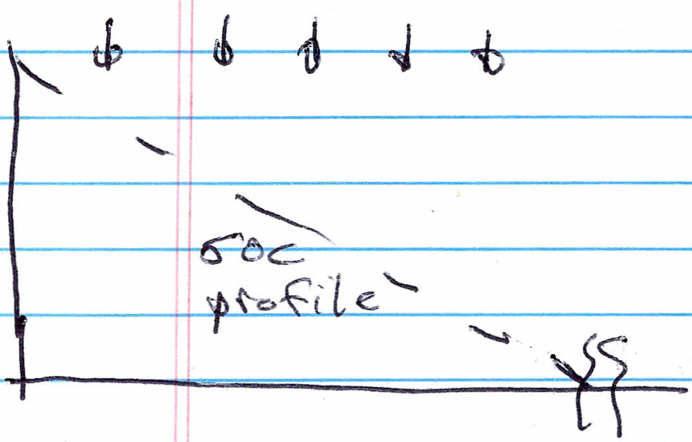
→ Basics

Consider:

→ box with ejecting bndry on RHS, accumulating bndry on LHS

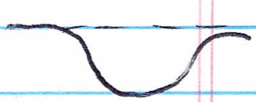
→ SOC profile, TBD

→ noise



→ Now, consider deviations from SOC profile, i.e.

 → bumps, blobs

 → voids, holes

→ Also assume conservation of "stuff" in the profile, up to boundary losses and noise source. Call stuff ρ.

→ Idea is to describe dynamics of deviation from SOC state

i.e. $\rho = \rho_{SOC} + \delta\rho$

ρ_{SOC} ↓ formally posited, not calculated.

∴ have

$$\boxed{\partial_t \delta\rho + \partial_x \left[\Gamma(\delta\rho) - D_0 \partial_x \delta\rho \right] = \tilde{S}}$$

- $\Gamma(\delta\rho)$ is flux induced by deviation from SOC state

- obviously, ρ conserved so $\delta\rho$ evolves via $\nabla \cdot \Gamma$ only

- background diffusion posited.

- $\Gamma(\delta\rho) \to 0$ as $\delta\rho \to 0$

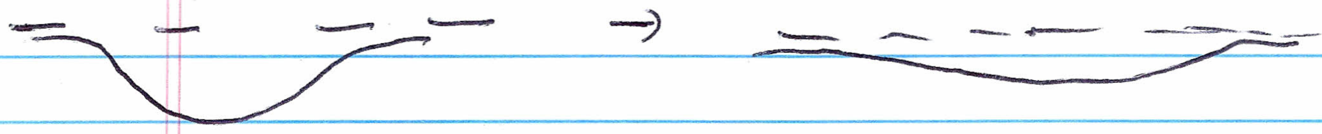
  $\delta\rho \to 0$ as $\tilde{S} \to 0$.

- How constrain $\Gamma(\delta\rho)$? $\to$ Symmetry!

  in spirit of Ginzburg / Landau prescription.

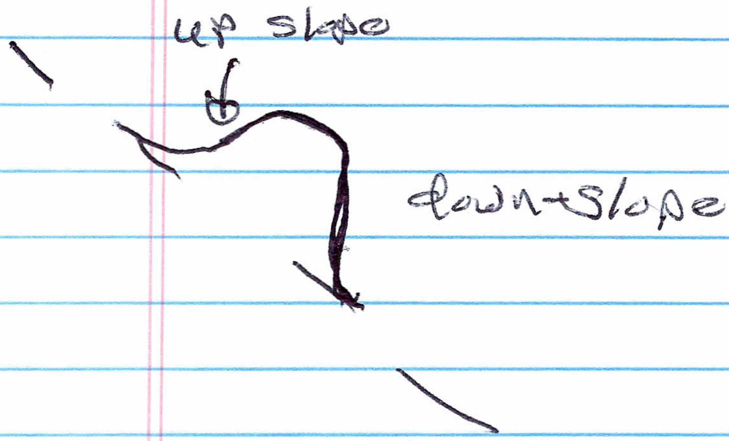
Now, consider

blob spreads out, conserving area.



likewise void.

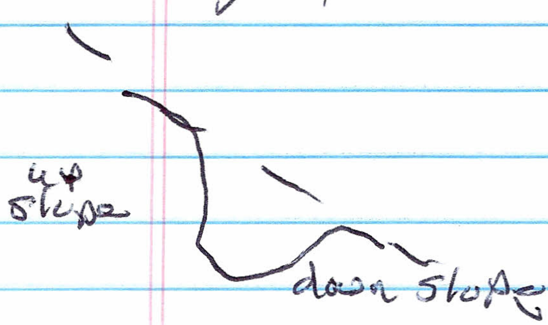
Now, if left-right symmetry broken by $\nabla \breve{p}_{sou} \neq 0$)



bumps ⇒ greater (extent / steepness) on down-slope

⇒ bumps / local excesses propagate <u>down</u> gradient, to <u>right</u>

Necessarily,



void ⇒ greater (extent / steepness) on up-slope than down slope

⇒ voids / local deficits propagate up gradient, to <u>left</u>

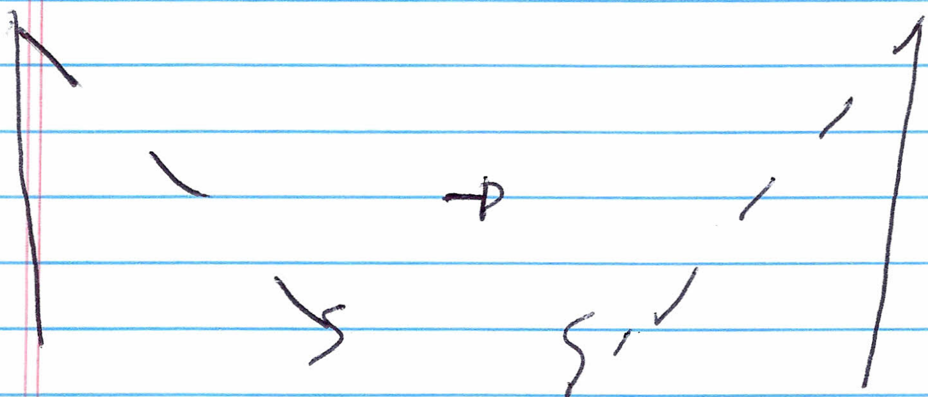
→ Both criteria local,

→ Both criteria common sense.

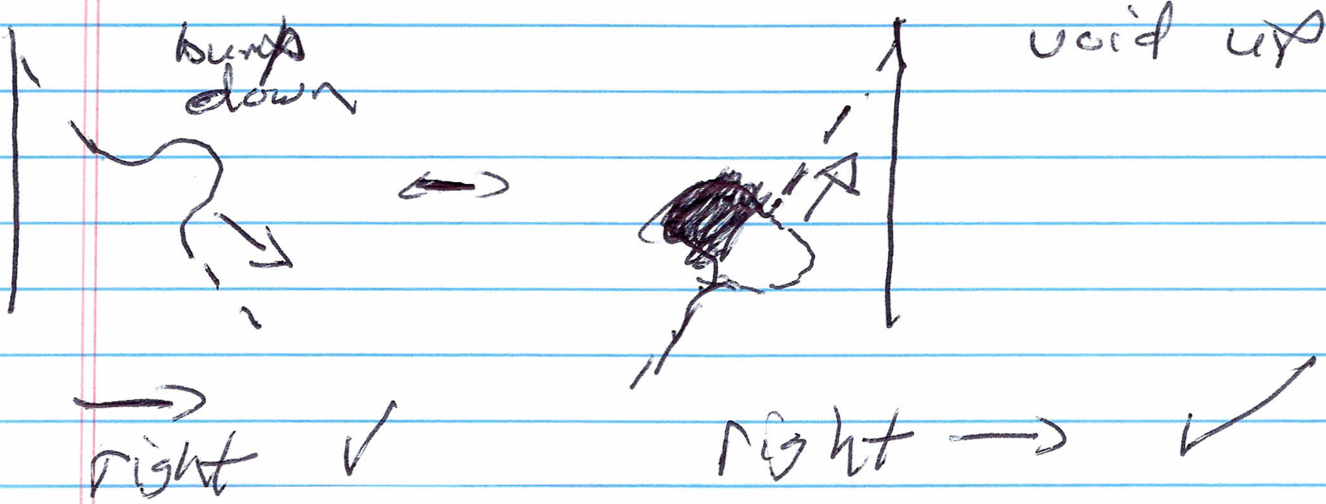
Now, observe

① reflection $x \to -y$

i.e.



② bump ⟷ hole interchange



same flux direction!

→ This brings us to the principle of joint reflection symmetry!

$$\Gamma\Big|_{\substack{x \to -x \\ \delta p \to -\delta p}} = \Gamma$$

This constrains the form of $\Gamma(\delta P)$!

How?

N.B.: - Full flux is complicated.

- seek flux in large scale, long time limit ⇒ smoothest form.

so have

$$\partial_t \delta P + \partial_x \left[\Gamma(\delta P) - D_0 \partial_x \delta P \right] = \tilde{S}$$

$\Gamma(\delta p)$ must satisfy joint reflection symmetry.

Then formally:

$$\Gamma(\delta P) = \sum_{\substack{m,n,\\ \varepsilon,r,\alpha}} \Big\{ A_n (\delta P)^n + B_m (\partial_x \delta P)^m + D_\alpha (\partial_x^2 \delta P)^\alpha + C_{\varepsilon,r} (\delta P)^\varepsilon (\partial_x \delta P)^r + \Big\}$$

JRS ≡ joint reflection symmetry.

1 $n=1$ violates JRS

1 $\simeq \alpha\, \delta P^2 + \text{h.o.t.}$

$\alpha > 0$

2 $m=1$ ok

$m=2$ ok

2 $\simeq -D\, \partial_x \delta P + \text{h.o.t.}$

$D > 0$ (well behaved)

3 $\alpha = 1$ violates JRS

$\alpha = 2$ too fine scaled

ignore.

(4) $q=1$, $r=1$ violates JRS

so, to lowest order in roughness:

$$\boxed{\partial_t \delta P + \partial_x\left[\alpha\, \delta P^2 - D\, \partial_x \delta P\right] - D_0\, \partial_x \delta P = \tilde{S}}$$

α, D are constants to be specified, as a, b in G-L theory are.

Re-absorb D_0 into D:

$$\boxed{\partial_t \delta P + \partial_x\left[\alpha\, \delta P^2 - D\, \partial_x \delta P\right] = \tilde{S}}$$

- hydro model limit is noisey Burgers
- exactly solvable for $\tilde{S}=0$
- basic solution structure is <u>shocks</u>!

- shocks akin avalanches, ...

Now, seek long wavelength approximation to nonlinear flux

i.e. $\left[\partial_x (\alpha\, \delta P^2)\right]_k \rightarrow d_k\, \delta P_k$

$\cong \gamma k^2\, \delta P_k$

γ → turbulent viscosity

N.B. $\alpha\, \delta P^2 \leftrightarrow D_T(\delta P)\, \delta P$

$\rightarrow -D(\nabla P - \nabla P_{crit})\, \nabla \delta P$

$\underbrace{\nabla P - \nabla P_{crit}}_{\delta P}$

clear correspondence to expected QL expression for flux, with threshold.

Now,

$$N_{k,\omega} = \left[\partial_x(\alpha\,\delta P^2)\right]_{k,\omega} \xrightarrow{\substack{\text{renormalized}\\ NL}} \gamma k^2 \delta P_{k,\omega}$$

$$= ik\alpha \sum_{k',\omega'} \delta P_{-k',-\omega'}\, \delta P_{k+k',\,\omega+\omega'}$$

$$\cong ik\alpha \sum_{k',\omega'} \delta P_{-k',-\omega}\, \delta P^{(2)}_{k+k',\,\omega+\omega'}$$

where:

nonlinear scrambling of coupling time

$$\left[-i(\omega+\omega') + (k+k')^2 D_0 + (k+k')^2 \gamma_k\right]$$

$$= -i\alpha(k+k')\,\delta P_{k',\omega'}\,\delta P_{k,\omega}$$

and substituting gives:

$$N_{k,\omega} = \gamma_T k^2 \delta P_{k,\omega}$$

where

for $k, \omega \to 0$; $\left\{ \begin{array}{l} \text{long, smooth} \\ \text{slow limit} \end{array} \right.$

$$\nu_T \cong \sum_{k', \omega'} \left| \delta P_{k', \omega'} \right|^2 \frac{k'^2 \nu_T}{\left[\omega'^2 + (k'^2 \nu_T)^2 \right]}$$

where neglected ν_0 relative to ν_T.

Now, need related $\delta P_{k', \omega'}$ to noise (i.e. $k', \omega' \to$ high freq, short wavelength modes excited). This must also include nonlinear response, self-consistently

so

$$(-i\omega' + k'^2 \nu)\, \delta P_{k', \omega'} = \tilde{S}_{k', \omega'}$$

so

$$\gamma = \alpha^2 \sum_{k',\omega'} \frac{|\tilde{s}_{k',\omega'}|^2}{(k'^2\gamma)^3} \, 1 \Big/ \left[1 + \left(\omega'/\gamma k'^2\right)^2\right]^2$$

$$\sum_{k',\omega'} = \int_{k_{min}}^{\infty} dk' \int d\omega'$$

and

$|\tilde{s}_{k,\omega}|^2 = S_0^2$ → white noise

⇒

$$\gamma = \frac{\overset{\#\,\text{const}}{\overset{\downarrow}{C_1}}\,\alpha^2 S_0^2}{\gamma^2} \int_{k_{min}}^{\infty} \frac{dk'}{k'^4}$$

→ infrared divergence!

→ why? ⇒
- conserved order parameter (flux form) $\partial_x \Gamma$
- slow modes

slow modes $\Rightarrow$ damping drops

$$\gamma \sim -k^2\nu$$

$$\Downarrow$$

$$\rightarrow 0 \text{ as } k \to 0$$

weak noise + tiny decay $\Rightarrow$
strong intensity.

$\Rightarrow$ general point: weakly damped modes dangerous, if any excitation available

So

$$\nu_T = \left(C_1 \alpha^2 S_0^2 \int_{k_{min}}^{\infty} \frac{dk'}{k'^4} \right)^{1/3}$$

$$\simeq \left(C_1' \alpha^2 S_0^2 \right)^{1/3} k_{min}^{-1}$$

$\Rightarrow$ ν_T depends explicitly on cut-off scale.

Now, meaning?

$k_{min}^{-1} \equiv \delta l \rightarrow$ scale being observed

$\delta l' < \delta l \Rightarrow$ scatterers

so

$\nu_T \sim \nu_{T,0}\, \delta l$ $\left\{ \begin{array}{l} \nu_T \text{ grows} \\ \text{with scale} \\ \text{of interest} \end{array} \right.$

but ν_T is diffusion $\Rightarrow$

$\frac{d}{dt}\langle \delta l^2 \rangle \sim \nu_T$, but

$\delta l^2 \sim \nu_{T_0}\, \delta l\, t$

$\Rightarrow \boxed{\delta l \sim \nu_{T,0}\, t} \Rightarrow \delta l$ pulse propagates ballistically not diffusively.

→ inferred divergence ultimately identifies ballistic propagation.

→ supported by scaling analysis

→ if 2D, anisotropic pile:

$$\partial_t \delta P + \partial_{\parallel}\left\{\alpha\, \delta P^2 - D\, \partial_{\parallel} \delta P\right\} = \underbrace{\nu_0 \nabla_{\perp} \delta P}_{=\tilde{S}}$$

$$\partial_{\parallel} = \frac{\nabla P}{|\nabla P|} \cdot \nabla$$ → derivative parallel to pile gradient of surface.

see refs for more.